2014

BOOKLET NO. TEST CODE : **UGA**

Forenoon

Questions : 30	**Time : 2 hours**

Write your Name, Registration Number, Test Centre, Test Code and the Number of this Booklet in the appropriate places on the Answersheet.

This test contains 30 questions in all. For each of the 30 questions, there are four suggested answers. Only one of the suggested answers is correct. You will have to identify the correct answer in order to get full credit for that question. Indicate your choice of the correct answer by darkening the appropriate oval ⬬, completely on the answersheet.

You will get
4 marks for each correctly answered question,
0 marks for each incorrectly answered question and
1 mark for each unattempted question.

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET ONLY.
YOU ARE NOT ALLOWED TO USE CALCULATOR.

WAIT FOR THE SIGNAL TO START.

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1. The system of inequalities

$$a - b^2 \geq \frac{1}{4},\ b - c^2 \geq \frac{1}{4},\ c - d^2 \geq \frac{1}{4},\ d - a^2 \geq \frac{1}{4} \qquad \text{has}$$

(A) no solutions (B) exactly one solution

(C) exactly two solutions (D) infinitely many solutions.

2. Let $\log_{12} 18 = a$. Then $\log_{24} 16$ is equal to

(A) $\dfrac{8-4a}{5-a}$ (B) $\dfrac{1}{3+a}$ (C) $\dfrac{4a-1}{2+3a}$ (D) $\dfrac{8-4a}{5+a}$.

3. The number of solutions of the equation $\tan x + \sec x = 2\cos x$, where $0 \leq x \leq \pi$, is

(A) 0 (B) 1 (C) 2 (D) 3.

4. Using only the digits 2, 3 and 9, how many six digit numbers can be formed which are divisible by 6?

(A) 41 (B) 80 (C) 81 (D) 161

5. What is the value of the following integral?

$$\int_{\frac{1}{2014}}^{2014} \frac{\tan^{-1} x}{x}\, dx$$

(A) $\dfrac{\pi}{4}\log 2014$ (B) $\dfrac{\pi}{2}\log 2014$ (C) $\pi \log 2014$ (D) $\dfrac{1}{2}\log 2014$

6. A light ray travelling along the line $y = 1$, is reflected by a mirror placed along the line $x = 2y$. The reflected ray travels along the line

(A) $4x - 3y = 5$ (B) $3x - 4y = 2$ (C) $x - y = 1$ (D) $2x - 3y = 1$.

7. For a real number x, let $[x]$ denote the greatest integer less than or equal to x. Then the number of real solutions of $\left| 2x - [x] \right| = 4$ is

(A) 1 (B) 2 (C) 3 (D) 4.

8. What is the ratio of the areas of the regular pentagons inscribed inside and circumscribed around a given circle?

(A) $\cos 36°$ (B) $\cos^2 36°$ (C) $\cos^2 54°$ (D) $\cos^2 72°$

9. Let z_1, z_2 be nonzero complex numbers satisfying $|z_1 + z_2| = |z_1 - z_2|$. The circumcentre of the triangle with the points z_1, z_2, and the origin as its vertices is given by

(A) $\frac{1}{2}(z_1 - z_2)$ (B) $\frac{1}{3}(z_1 + z_2)$ (C) $\frac{1}{2}(z_1 + z_2)$ (D) $\frac{1}{3}(z_1 - z_2)$.

10. In how many ways can 20 identical chocolates be distributed among 8 students so that each student gets at least one chocolate and exactly two students get at least two chocolates each?

(A) 308 (B) 364 (C) 616 (D) $\binom{8}{2}\binom{17}{7}$

11. Two vertices of a square lie on a circle of radius r, and the other two vertices lie on a tangent to this circle. Then, each side of the square is

(A) $\frac{3r}{2}$ (B) $\frac{4r}{3}$ (C) $\frac{6r}{5}$ (D) $\frac{8r}{5}$.

12. Let P be the set of all numbers obtained by multiplying five distinct integers between 1 and 100. What is the largest integer n such that 2^n divides at least one element of P?

(A) 8 (B) 20 (C) 24 (D) 25

13. Consider the function $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d are real numbers with $a > 0$. If f is strictly increasing, then the function $g(x) = f'(x) - f''(x) + f'''(x)$ is

(A) zero for some $x \in \mathbb{R}$ (B) positive for all $x \in \mathbb{R}$

(C) negative for all $x \in \mathbb{R}$ (D) strictly increasing.

14. Let A be the set of all points (h, k) such that the area of the triangle formed by (h, k), $(5, 6)$ and $(3, 2)$ is 12 square units. What is the least possible length of a line segment joining $(0, 0)$ to a point in A?

(A) $\frac{4}{\sqrt{5}}$ (B) $\frac{8}{\sqrt{5}}$ (C) $\frac{12}{\sqrt{5}}$ (D) $\frac{16}{\sqrt{5}}$

15. Let $P = \{abc : a, b, c \text{ positive integers}, \ a^2 + b^2 = c^2, \text{ and 3 divides } c\}$. What is the largest integer n such that 3^n divides every element of P?

(A) 1 (B) 2 (C) 3 (D) 4

16. Let $A_0 = \emptyset$ (the empty set). For each $i = 1, 2, 3, \ldots$, define the set $A_i = A_{i-1} \cup \{A_{i-1}\}$. The set A_3 is

(A) $\emptyset$ (B) $\{\emptyset\}$ (C) $\{\emptyset, \{\emptyset\}\}$ (D) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

17. Let $f(x) = \dfrac{1}{x-2}\cdot$ The graphs of the functions f and f^{-1} intersect at

(A) $(1+\sqrt{2}, 1+\sqrt{2})$ and $(1-\sqrt{2}, 1-\sqrt{2})$

(B) $(1+\sqrt{2}, 1+\sqrt{2})$ and $(\sqrt{2}, -1-\frac{1}{\sqrt{2}})$

(C) $(1-\sqrt{2}, 1-\sqrt{2})$ and $(-\sqrt{2}, -1+\frac{1}{\sqrt{2}})$

(D) $(\sqrt{2}, -1-\frac{1}{\sqrt{2}})$ and $(-\sqrt{2}, -1+\frac{1}{\sqrt{2}})$

18. Let N be a number such that whenever you take N consecutive positive integers, at least one of them is coprime to 374. What is the smallest possible value of N?

(A) 4 (B) 5 (C) 6 (D) 7

19. Let $A_1, A_2, \ldots, A_{18}$ be the vertices of a regular polygon with 18 sides. How many of the triangles $\triangle A_iA_jA_k$, $1 \le i < j < k \le 18$, are isosceles but not equilateral?

(A) 63 (B) 70 (C) 126 (D) 144

20. The limit $\displaystyle\lim_{x\to 0} \frac{\sin^\alpha x}{x}$ exists *only when*

(A) $\alpha \ge 1$ (B) $\alpha = 1$

(C) $|\alpha| \le 1$ (D) α is a positive integer.

21. Consider the region $R = \{(x, y) : x^2 + y^2 \le 100, \ \sin(x+y) > 0\}$. What is the area of R?

(A) 25π (B) 50π (C) 50 (D) $100\pi - 50$

22. Consider a cyclic trapezium whose circumcentre is on one of the sides. If the ratio of the two parallel sides is 1 : 4, what is the ratio of the sum of the two oblique sides to the longer parallel side?

(A) $\sqrt{3} : \sqrt{2}$ (B) $3 : 2$ (C) $\sqrt{2} : 1$ (D) $\sqrt{5} : \sqrt{3}$

23. Consider the function $f(x) = \left\{ \log_e \left(\frac{4 + \sqrt{2x}}{x} \right) \right\}^2$ for $x > 0$. Then,

(A) f decreases upto some point and increases after that

(B) f increases upto some point and decreases after that

(C) f increases initially, then decreases and then again increases

(D) f decreases initially, then increases and then again decreases.

24. What is the number of ordered triplets (a, b, c), where a, b, c are positive integers (not necessarily distinct), such that $abc = 1000$?

(A) 64 (B) 100 (C) 200 (D) 560

25. Let $f : (0, \infty) \to (0, \infty)$ be a function differentiable at 3, and satisfying $f(3) = 3f'(3) > 0$. Then the limit

$$\lim_{x \to \infty} \left(\frac{f\left(3 + \frac{3}{x}\right)}{f(3)} \right)^x$$

(A) exists and is equal to 3 (B) exists and is equal to e

(C) exists and is always equal to $f(3)$ (D) need not always exist.

26. Let z be a non-zero complex number such that $\left| z - \frac{1}{z} \right| = 2$. What is the maximum value of $|z|$?

(A) 1 (B) $\sqrt{2}$ (C) 2 (D) $1 + \sqrt{2}$.

27. The minimum value of

$$\left| \sin x + \cos x + \tan x + \operatorname{cosec} x + \sec x + \cot x \right| \quad \text{is}$$

(A) 0 (B) $2\sqrt{2} - 1$ (C) $2\sqrt{2} + 1$ (D) 6

28. For any function $f : X \to Y$ and any subset A of Y, define

$$f^{-1}(A) = \{x \in X : f(x) \in A\}.$$

Let A^c denote the complement of A in Y. For subsets A_1, A_2 of Y, consider the following statements:

(i) $f^{-1}(A_1^c \cap A_2^c) = (f^{-1}(A_1))^c \cup (f^{-1}(A_2))^c$
(ii) If $f^{-1}(A_1) = f^{-1}(A_2)$ then $A_1 = A_2$.

Then,

(A) both (i) and (ii) are always true

(B) (i) is always true, but (ii) may not always be true

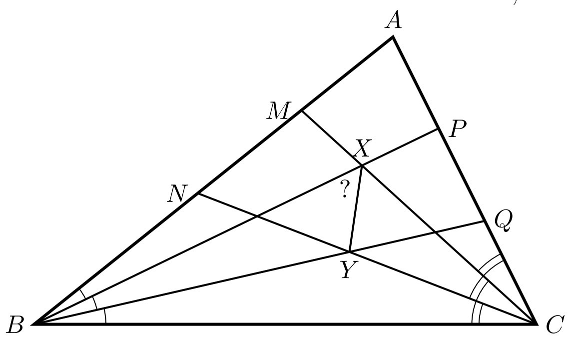
(C) (ii) is always true, but (i) may not always be true

(D) neither (i) nor (ii) is always true.

29. Let f be a function such that $f''(x)$ exists, and $f''(x) > 0$ for all $x \in [a, b]$. For any point $c \in [a, b]$, let $A(c)$ denote the area of the region bounded by $y = f(x)$, the tangent to the graph of f at $x = c$ and the lines $x = a$ and $x = b$. Then

(A) $A(c)$ attains its minimum at $c = \frac{1}{2}(a + b)$ for any such f
(B) $A(c)$ attains its maximum at $c = \frac{1}{2}(a + b)$ for any such f
(C) $A(c)$ attains its minimum at both $c = a$ and $c = b$ for any such f
(D) the points c where $A(c)$ attains its minimum depend on f.

30. In $\triangle ABC$, the lines BP, BQ trisect $\angle ABC$ and the lines CM, CN trisect $\angle ACB$. Let BP and CM intersect at X and BQ and CN intersect at Y. If $\angle ABC = 45°$ and $\angle ACB = 75°$, then $\angle BXY$ is



(A) 45° (B) $47\frac{1}{2}°$ (C) 50° (D) 55°