2015

BOOKLET No. TEST CODE : **UGB**

Afternoon

Answer all questions	**Time : 2 hours**

Write the Test Code, Test Booklet number, your name, Registration number, Test Code and the Number of this Booklet in the appropriate places on the answer booklet.

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET AND/OR ON THE ANSWER BOOKLET. YOU ARE NOT ALLOWED TO USE A CALCULATOR.

Answer to each question should start on a fresh page

STOP! WAIT FOR THE SIGNAL TO START.

1. Let $0 < a_1 < a_2 < \cdots < a_n$ be real numbers. Show that the equation

$$\frac{a_1}{a_1 - x} + \frac{a_2}{a_2 - x} + \cdots + \frac{a_n}{a_n - x} = 2015$$

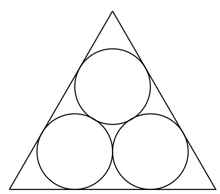
has exactly n real roots.

2. Let $\mathbb{R}$ denote the set of real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$, satisfying

$$|f(x) - f(y)| = 2|x - y|$$

for all $x, y \in \mathbb{R}$. Justify your answer.

3. Three circles of unit radius tangentially touch each other in the plane. Consider the triangle enclosing them such that each side of the triangle is tangential to two of these three circles. See picture below:



Find the length of each side of the triangle.

4. Let a and b be real numbers. Define a function $f : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers, by the formula $f(x) = x^2 + ax + b$. Assume that the graph of f intersects the co-ordinate axes in three distinct points. Prove that the circle passing through these three points also passes through the point $(0, 1)$.

5. Find all positive integers n for which $5^n + 1$ is divisible by 7. Justify your answer.

6. Let $p(x) = x^7 + x^6 + b_5x^5 + \cdots + b_1x + b_0$ and $q(x) = x^5 + c_4x^4 + \cdots + c_1x + c_0$ be polynomials with integer coefficients. Assume that $p(i) = q(i)$ for integers $i = 1, 2, \ldots, 6$. Then, show that there exists a negative integer r such that $p(r) = q(r)$.

[P. T. O.]

7. Let $S = \{1, 2, \ldots, j\}$. For every non-empty subset A of S, let $m(A)$ denote the maximum element of A. Then, show that

$$\sum m(A) = (j-1)2^j + 1$$

where the summation in the left hand side of the above equation is taken over all non-empty subsets A of S.

8. (a) Let $m_1 < m_2 < \cdots < m_k$ be positive integers such that $\frac{1}{m_1}, \frac{1}{m_2}, \cdots, \frac{1}{m_k}$ are in arithmetic progression. Then prove that $k < m_1 + 2$.

   (b) For any integer $k > 0$, give an example of a sequence of k positive integers whose reciprocals are in arithmetic progression.