B.Math/B.Stat ENTRANCE EXAM 2016

BOOKLET No. TEST CODE : UGB

Afternoon Session

Answer as many as you can

Time : 2 hours

Write your Name, Registration number, Test Centre, Test Code and the Number of this booklet in the appropriate places on the answer-booklet.

ALL ROUGH WORK IS TO BE DONE ON THIS BOOKLET
AND/OR THE ANSWER-BOOKLET.
CALCULATORS ARE NOT ALLOWED.

STOP! WAIT FOR THE SIGNAL TO START.

Q 1. Suppose that in a sports tournament featuring n players, each pair plays one game and there is always a winner and a loser (no draws). Show that the players can be arranged in an order $P_1, P_2, \cdots, P_n$ such that player P_i has beaten P_{i+1} for all $i = 1, 2, \cdots, n-1$.

Q 2. Consider a cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ where a, b, c, d are integers such that ad is odd and bc is even. Prove that not all roots of $p(x)$ can be rational.

Q 3. Given the polynomial

$$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n,$$

with real coefficients, and $a_1^2 < a_2$, show that not all roots of $f(x)$ can be real.

Q 4. Let d be a positive integer. Prove that there exists a right-angled triangle with rational sides and area equal to d if and only if there exists an arithmetic progression x^2, y^2, z^2 of squares of rational numbers whose common difference is d.

Q 5. Let $ABCD$ be a square two of whose adjacent vertices, say A, B, are on the positive X-axis and the positive Y-axis, respectively. If C has co-ordinates (u, v) in the first quadrant, determine the area of $ABCD$ in terms of u and v.

Q 6. Let a, b, c be the sides of a triangle and A, B, C be the angles opposite to these sides respectively. If

$$\sin(A - B) = \frac{a}{a+b}\sin A\cos B - \frac{b}{a+b}\cos A\sin B,$$

then prove that the triangle is isosceles.

Q 7. Let f be a differentiable function such that $f(f(x)) = x$ for $x \in [0, 1]$. Suppose $f(0) = 1$. Determine the value of $\int_0^1 (x - f(x))^{2016}\,\mathrm{dx}$.

Q 8. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers defined recursively by

$$a_{n+1} = \frac{3a_n}{2 + a_n},$$

for all $n \geq 1$.

(i) If $0 < a_1 < 1$, then prove that $\{a_n\}_{n\geq 1}$ is increasing and $\lim_{n\to\infty} a_n = 1$.

(ii) If $a_1 > 1$, then prove that $\{a_n\}_{n\geq 1}$ is decreasing and $\lim_{n\to\infty} a_n = 1$.