Correct answers are ticked in green.

MTH

GS-2015 (Mathematics)

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Written Test in **MATHEMATICS - December 14, 2014**
Duration : Two hours (2 hours)

Name : ____ Ref. Code : ____________

Please read all instructions carefully before you attempt the questions.

1. Please fill in details about name, reference code etc. on the answer sheet. The Answer Sheet is machine-readable. Use only Black/Blue ball point pen to fill in the answer sheet.

2. There are thirty (30) multiple choice questions divided into two parts. Part I consists of 15 questions and Part II consists of 15 questions. Bachelors students who have applied only for the Integrated Ph.D. program at TIFR CAM, Bangalore will only be evaluated on Part I. All other students (including Bachelors students applying to the Ph.D. programs at both TIFR, Mumbai and Bangalore) will be evaluated on both Parts I and II.

3. Indicate your answer ON THE ANSWER SHEET by blackening the appropriate circle for each question. **Each corect answer will get 1 mark. There is no negative marking for wrong answers. A question not answered will not get you any mark.** Do not mark more than one circle for any question : this will be treated as a wrong answer.

4. We advise you to first mark the correct answers on the QUESTION PAPER and then to TRANSFER these to the ANSWER SHEET only when you are sure of your choice.

5. Rough work may be done on blank pages of the question paper. If needed, you may ask for extra rough sheets from an Invigilator.

6. **Use of calculators is NOT permitted.**

7. Do NOT ask for clarifications from the invigilators regarding the questions. They have been instructed not to respond to any such inquiries from candidates. In case a correction/clarification is deemed necessary, the invigilators will announce it publicly.

8. Notation and Conventions used in this test are given on page 2 of the question paper.

NOTATION AND CONVENTIONS

$\mathbb{N}$:= Set of natural numbers = $\{1, 2, 3, \ldots\}$
$\mathbb{Z}$:= Set of integers
$\mathbb{Q}$:= Set of rational numbers
$\mathbb{R}$:= Set of real numbers
$\mathbb{C}$:= Set of complex numbers
$\mathbb{R}^n$:= n-dimensional vector space over $\mathbb{R}$

$(a, b) := \{x \in \mathbb{R} | a < x < b\}$
$(a, b] := \{x \in \mathbb{R} | a < x \leq b\}$
$[a, b) := \{x \in \mathbb{R} | a \leq x < b\}$
$[a, b] := \{x \in \mathbb{R} | a \leq x \leq b\}$

A sequence is always indexed by the set of natural numbers.
The cyclic group with n elements is denoted by $\mathbb{Z}_n$.
Unless stated otherwise, subsets of $\mathbb{R}^n$ carry the induced topology.
For any set S, the cardinality of S is denoted by $|S|$.

Part I

1. Let A be an invertible 10×10 matrix with real entries such that the sum of each row is 1. Then

   A. The sum of the entries of each row of the inverse of A is 1
   B. The sum of the entries of each column of the inverse of A is 1
   C. The trace of the inverse of A is non-zero
   D. None of the above.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Which one of the following sets cannot be the image of $(0, 1]$ under f ?

   A. $\{0\}$
   B. $(0, 1)$
   C. $[0, 1)$
   D. $[0, 1]$.

3. Let A be a 10×10 matrix with complex entries such that all its eigenvalues are non-negative real numbers, and at least one eigenvalue is positive. Which of the following statements is always false ?

   A. There exists a matrix B such that $AB - BA = B$
   B. There exists a matrix B such that $AB - BA = A$
   C. There exists a matrix B such that $AB + BA = A$
   D. There exists a matrix B such that $AB + BA = B$.

4. Let S be the collection of (isomorphism classes of) groups G which have the property that every element of G commutes only with the identity element and itself. Then

A. $|S| = 1$
B. $|S| = 2$
C. $|S| \geq 3$ and is finite
D. $|S| = \infty$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ denote the function defined by $f(x) = (1-x^2)^{\frac{3}{2}}$ if $|x| < 1$, and $f(x) = 0$ if $|x| \geq 1$. Which of the following statements is correct ?

A. f is not continuous
B. f is continuous but not differentiable
C. f is differentiable but $f^{'}$ is not continuous
D. f is differentiable and $f^{'}$ is continuous.

6. Let A be the 2×2 matrix $\begin{pmatrix} \sin\frac{\pi}{18} & -\sin\frac{4\pi}{9} \\ \sin\frac{4\pi}{9} & \sin\frac{\pi}{18} \end{pmatrix}$. Then the smallest number $n \in \mathbb{N}$ such that $A^n = I$ is

A. 3
B. 9
C. 18
D. 27.

7. Let f and g be two functions from $[0, 1]$ to $[0, 1]$ with f strictly increasing. Which of the following statements is always correct ?

A. If g is continuous, then $f \circ g$ is continuous
B. If f is continuous, then $f \circ g$ is continuous
C. If f and $f \circ g$ are continuous, then g is continuous
D. If g and $f \circ g$ are continuous, then f is continuous.

8. Let $f(x) = \frac{e^{-\frac{1}{x}}}{x}$, where $x \in (0,1)$. Then, on $(0,1)$

A. f is uniformly continuous
B. f is continuous but not uniformly continuous
C. f is unbounded
D. f is not continuous.

9. Let $\{a_n\}$ be a sequence of real numbers such that $|a_{n+1} - a_n| \leq \frac{n^2}{2^n}$ for all $n \in \mathbb{N}$. Then

A. The sequence $\{a_n\}$ may be unbounded
B. The sequence $\{a_n\}$ is bounded but may not converge
C. The sequence $\{a_n\}$ has exactly two limit points
D. The sequence $\{a_n\}$ is convergent.

10. For a group G, let $\mathrm{Aut}(G)$ denote the group of automorphisms of G. Which of the following statements is true ?

A. $\mathrm{Aut}(\mathbb{Z})$ is isomorphic to $\mathbb{Z}_2$
B. If G is cyclic, then $\mathrm{Aut}(G)$ is cyclic
C. If $\mathrm{Aut}(G)$ is trivial, then G is trivial
D. $\mathrm{Aut}(\mathbb{Z})$ is isomorphic to $\mathbb{Z}$.

11. Let $\{a_n\}$ be a sequence of real numbers. Which of the following is true ?

A. If $\sum a_n$ converges, then so does $\sum a_n^4$
B. If $\sum |a_n|$ converges, then so does $\sum a_n^2$
C. If $\sum a_n$ diverges, then so does $\sum a_n^3$
D. If $\sum |a_n|$ diverges, then so does $\sum a_n^2$.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function that vanishes at 10 distinct points in $\mathbb{R}$. Suppose $f^{(n)}$ denotes the n-th derivative of f, for $n \geq 1$. Which of the following statements is always true ?

A. $f^{(n)}$ has at least 10 zeros, for $1 \leq n \leq 8$
B. $f^{(n)}$ has at least one zero, for $1 \leq n \leq 9$
C. $f^{(n)}$ has at least 10 zeros, for $n \geq 10$
D. $f^{(n)}$ has at least one zero, for $n \geq 9$.

13. For a real number $t > 0$, let $\sqrt{t}$ denote the positive square root of t. For a real number $x > 0$, let $F(x) = \int_{x^2}^{4x^2} \sin\sqrt{t}\, dt$. If F' is the derivative of F, then

A. $F'(\frac{\pi}{2}) = 0$
B. $F'(\frac{\pi}{2}) = \pi$
C. $F'(\frac{\pi}{2}) = -\pi$
D. $F'(\frac{\pi}{2}) = 2\pi$.

14. Let $n \in \mathbb{N}$ be a six digit number whose base 10 expansion is of the form $abcabc$, where a, b, c are digits between 0 and 9 and a is non-zero. Then

A. n is divisible by 5
B. n is divisible by 8
C. n is divisible by 13
D. n is divisible by 17.

15. The series $\sum_{n=1}^{\infty} \frac{\cos(3^n x)}{2^n}$

A. Diverges, for all rational $x \in \mathbb{R}$
B. Diverges, for some irrational $x \in \mathbb{R}$
C. Converges, for some but not all $x \in \mathbb{R}$
D. Converges, for all $x \in \mathbb{R}$.

Part II

16. Let X be a proper closed subset of $[0,1]$. Which of the following statements is always true ?

    A. The set X is countable
    B. There exists $x \in X$ such that $X \setminus \{x\}$ is closed
    C. The set X contains an open interval
    D. None of the above. ✓

17. In how many ways can the group $\mathbb{Z}_5$ act on the set $\{1,2,3,4,5\}$?

    A. 5
    B. 24
    C. 25 ✓
    D. 120.

18. Let f be a function from $\{1,2,\ldots,10\}$ to $\mathbb{R}$ such that

$$\left(\sum_{i=1}^{10} \frac{|f(i)|}{2^i}\right)^2 = \left(\sum_{i=1}^{10} |f(i)|^2\right)\left(\sum_{i=1}^{10} \frac{1}{4^i}\right).$$

Mark the correct statement.

    A. There are uncountably many f with this property ✓
    B. There are only countably infinitely many f with this property
    C. There is exactly one such f
    D. There is no such f.

19. Let $U_1 \supset U_2 \supset \cdots$ be a decreasing sequence of open sets in Euclidean 3-space $\mathbb{R}^3$. What can we say about the set $\cap U_i$?

A. It is infinite
B. It is open
C. It is non-empty
D. None of the above.

20. Let $n \geq 1$ and let A be an $n \times n$ matrix with real entries such that $A^k = 0$, for some $k \geq 1$. Let I be the identity $n \times n$ matrix. Then

A. $I + A$ need not be invertible
B. $\mathrm{Det}(I + A)$ can be any non-zero real number
C. $\mathrm{Det}(I + A) = 1$
D. A^n is a non-zero matrix.

21. Let $f : [0, 1] \to \mathbb{R}$ be a fixed continuous function such that f is differentiable on $(0, 1)$ and $f(0) = f(1) = 0$. Then the equation $f(x) = f'(x)$ admits

A. No solution $x \in (0, 1)$
B. More than one solution $x \in (0, 1)$
C. Exactly one solution $x \in (0, 1)$
D. At least one solution $x \in (0, 1)$.

22. A complex number $\alpha \in \mathbb{C}$ is called *algebraic* if there is a non-zero polynomial $P(x) \in \mathbb{Q}[x]$ with rational coefficients such that $P(\alpha) = 0$. Which of the following statements is true ?

A. There are only finitely many algebraic numbers
B. All complex numbers are algebraic
C. $\sin(\frac{\pi}{3}) + \cos(\frac{\pi}{4})$ is algebraic
D. None of the above.

23. For a group G, let $F(G)$ denote the collection of all subgroups of G. Which one of the following situations can occur ?

A. G is finite but $F(G)$ is infinite
B. G is infinite but $F(G)$ is finite
C. G is countable but $F(G)$ is uncountable ✓
D. G is uncountable but $F(G)$ is countable.

24. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $A \subset \mathbb{R}$ be defined by

$$A = \{y \in \mathbb{R} : y = \lim_{n\to\infty} f(x_n), \text{ for some sequence } x_n \to +\infty\}.$$

Then the set A is necessarily

A. A connected set ✓
B. A compact set
C. A singleton set
D. None of the above.

25. How many finite sequences $x_1, x_2, \ldots, x_m$ are there such that each $x_i = 1$ or 2, and $\sum_{i=1}^{m} x_i = 10$?

A. 89 ✓
B. 91
C. 92
D. 120.

26. Let (X, d) be a path connected metric space with at least two elements, and let $S = \{d(x, y) : x, y \in X\}$. Which of the following statements is not necessarily true ?

A. S is infinite
B. S contains a non-zero rational number
C. S is connected
D. S is a closed subset of $\mathbb{R}$. ✓

27. Let $X \subset \mathbb{R}$ and let $f, g : X \to X$ be continuous functions such that $f(X) \cap g(X) = \emptyset$ and $f(X) \cup g(X) = X$. Which one of the following sets cannot be equal to X ?

A. $[0, 1]$ ✓
B. $(0, 1)$
C. $[0, 1)$
D. $\mathbb{R}$.

28. Let $X = \{(x, y) \in \mathbb{R}^2 : 2x^2 + 3y^2 = 1\}$. Endow $\mathbb{R}^2$ with the discrete topology, and X with the subspace topology. Then

A. X is a compact subset of $\mathbb{R}^2$ in this topology
B. X is a connected subset of $\mathbb{R}^2$ in this topology
C. X is an open subset of $\mathbb{R}^2$ in this topology ✓
D. None of the above.

29. Let G be a group. Suppose $|G| = p^2q$, where p and q are distinct prime numbers satisfying $q \not\equiv 1 \mod p$. Which of the following is always true ?

A. G has more than one p-Sylow subgroup
B. G has a normal p-Sylow subgroup ✓
C. The number of q-Sylow subgroups of G is divisible by p
D. G has a unique q-Sylow subgroup.

30. Let $d(x, y)$ be the usual Euclidean metric on $\mathbb{R}^2$. Which of the following metric spaces is complete ?

A. $\mathbb{Q}^2 \subset \mathbb{R}^2$ with the metric $d(x, y)$
B. $[0, 1] \times [0, \infty) \subset \mathbb{R}^2$ with the metric $d'(x, y) = \frac{d(x,y)}{1+d(x,y)}$ ✓
C. $(0, \infty) \times [0, \infty) \subset \mathbb{R}^2$ with the metric $d(x, y)$
D. $[0, 1] \times [0, 1) \subset \mathbb{R}^2$ with the metric $d''(x, y) = \min\{1, d(x, y)\}$.