2016

BOOKLET No. TEST CODE: PSB

Afternoon

Questions: 10	**Time: 2 hours**

Write your Registration number, Test Centre, Test Code, and the Number of this booklet in the appropriate places on the answer-book.

- All questions carry equal weight.
- Answer two questions from GROUP A and four questions from GROUP B.
- Best six answers subject to the above conditions will be considered.

Answer to each question should start on a fresh page.

All rough work must be done on this booklet and/or the answer-book. You are not allowed to use calculators in any form.

STOP! WAIT FOR THE SIGNAL TO START.

GROUP A

1. Let x, y be real numbers such that $xy = 10$. Find the minimum value of $|x+y|$ and all also find all the points (x, y) where this minimum value is achieved.

   Justify your answer.

2. Determine the average value of

$$i_1 i_2 + i_2 i_3 + \cdots + i_9 i_{10} + i_{10} i_1$$

   taken over all permutations $i_1, i_2, \ldots, i_{10}$ of $1, 2, \ldots, 10$.

3. For any two events A and B, show that

$$(\mathrm{P}(A \cap B))^2 + (\mathrm{P}(A \cap B^c))^2 + (\mathrm{P}(A^c \cap B))^2 + (\mathrm{P}(A^c \cap B^c))^2 \geq \frac{1}{4}.$$

4. Let X, Y, and Z be three Bernoulli $(\frac{1}{2})$ random variables such that X and Y are independent, Y and Z are independent, and Z and X are independent.

   (a) Show that $\mathrm{P}(XYZ = 0) \geq \frac{3}{4}$.

   (b) Show that if equality holds in (a), then $Z = \begin{cases} 1 & \text{if } X = Y, \\ 0 & \text{if } X \neq Y. \end{cases}$

GROUP B

5. Let $n \geq 2$, and $X_1, X_2, \ldots, X_n$ be independent and identically distributed Poisson (λ) random variables for some $\lambda > 0$. Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ denote the corresponding order statistics.

   (a) Show that $\mathrm{P}(X_{(2)} = 0) \geq 1 - n(1 - e^{-\lambda})^{n-1}$.

   (b) Evaluate the limit of $\mathrm{P}(X_{(2)} > 0)$ as the sample size $n \to \infty$.

6. Suppose that random variables X and Y jointly have a bivariate normal distribution with $\mathrm{E}(X) = \mathrm{E}(Y) = 0, \mathrm{Var}(X) = \mathrm{Var}(Y) = 1$, and correlation ρ. Compute the correlation between e^X and e^Y.

7. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with probability mass function

$$f(x;\theta) = \frac{x\theta^x}{h(\theta)} \qquad \text{for } x = 1, 2, 3, \ldots$$

where $0 < \theta < 1$ is an unknown parameter and $h(\theta)$ is a function of θ. Show that the maximum likelihood estimator of θ is also a method of moments estimator.

8. Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be independent and identically distributed pairs of random variables with $\mathrm{E}(X_1) = \mathrm{E}(Y_1), \mathrm{Var}(X_1) = \mathrm{Var}(Y_1) = 1$, and $\mathrm{Cov}(X_1, Y_1) = \rho \in (-1, 1)$.

   (a) Show that there exists a function $c(\rho)$ such that

$$\lim_{n\to\infty} \mathrm{P}\left(\sqrt{n}(\overline{X} - \overline{Y}) \leq c(\rho)\right) = \Phi(1)$$

   where Φ denotes the standard normal cumulative distribution function.

   (b) Given $\alpha \in (0, 1)$, obtain a statistic L_n which is a function of $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ such that

$$\lim_{n\to\infty} \mathrm{P}(L_n < \rho < 1) = \alpha.$$

9. Suppose $X_1, X_2, \ldots, X_N$ are independent exponentially distributed random variables with mean 1, where N is unknown. We only observe the largest X_i value and denote it by T. We want to test $H_0 : N = 5$ against $H_1 : N = 10$. Show that the most powerful test of size 0.05 rejects H_0 when $T > c$ for some c, and determine c.

10. Consider a population with $N > 1$ units having values $y_1, y_2, \ldots, y_N$. A sample of size n_1 is drawn from the population using SRSWOR. From the remaining part of the population, a sample of size n_2 is drawn using SRSWOR. Show that the covariance between the two sample means is

$$\frac{-\sum_{i=1}^{N}(y_i - \bar{y})^2}{N(N-1)},$$

where $\bar{y} = \dfrac{1}{N}\sum_{i=1}^{N} y_i$.